Appendix A

Derivation of mutual inductance between two dipole coils

Here, we derive the mutual inductance between two magnetic dipole coils:

$$L_m = \frac{\mu_o AB}{4\pi R^3}\left(3(\widehat{a}\cdot\widehat{r})(\widehat{r}\cdot\widehat{b}) - \widehat{a}\cdot\widehat{b}\right) \tag{A.1}$$

where
$\widehat{a}$ = unit vector in direction of dipole A
A = magnitude of effective area of dipole A
$\widehat{b}$ = unit vector in direction of dipole B
B = magnitude of effective area of dipole B
$\widehat{r}$ = unit vector in direction from center of dipole A to center of dipole B
R = magnitude of distance from center of dipole A to center of dipole B

Each dipole is replaced by a small but finite-sized single-turn square coil. Each coil is shrunk to an infinitesimal size (while increasing its number of turns to infinity) to obtain the dipole limit. In the dipole limit, the shape of the coil does not matter. The coil's properties are described by the coil's effective area, the product of the coil's geometrical area and number of turns.

The mutual inductance, L_m, between two closed circuits is given by Neuman's formula:

$$L_m = \frac{\mu_o}{4\pi}\oint_{circuit A}\oint_{circuit B}\frac{\overline{ds_A}\cdot\overline{ds_B}}{r_{AB}} \tag{A.2}$$

where
r_{AB} = distance between point on circuit A and point on circuit$B$
$\mu_o = 4\pi 10^{-7} henries/meter$
Here, we consider the case where each circuit is composed of four straight-line segments.
Define some quantities:
$\overline{A} = A\widehat{a}$ = effective area of dipole A
$\widehat{u}$ and $\widehat{v}$ are two unit vectors such that $\widehat{a} = \widehat{u}\times\widehat{v}$
$\overline{B} = B\widehat{b}$ = effective area of dipole B
$\widehat{p}$ and $\widehat{q}$ are two unit vectors such that $\widehat{b} = \widehat{p}\times\widehat{q}$
h = a small dimensionless scalar which goes to zero in the dipole limit
$\overline{R} = R\widehat{r}$ = vector from center of dipole A to center of dipole B

Use the $\widehat{u}$, $\widehat{v}$, $\widehat{a}$ cartesian coordinate system, where dipole A is at the origin and dipole B is at $\overline{R} = R\widehat{r} = u\widehat{u} + v\widehat{v} + a\widehat{a}$.

The four corners of coil A are:

$\overline{A_1} = \frac{h}{2}\widehat{u} + \frac{h}{2}\widehat{v}$

$\overline{A_2} = \frac{h}{2}\widehat{u} - \frac{h}{2}\widehat{v}$

$\overline{A_3} = -\frac{h}{2}\widehat{u} - \frac{h}{2}\widehat{v}$

$\overline{A_4} = -\frac{h}{2}\widehat{u} + \frac{h}{2}\widehat{v}$

Note that the ratio of the effective area of dipole A to the effective area of coil A is $\frac{A}{h^2}$.

A point on coil A is given by:

$(\mu, \nu) = \mu\,\widehat{u} + \nu\,\widehat{v}$

where μ has nothing to do with μ_o.

The four corners of coil B are:

$\overline{B_1} = \frac{h}{2}\widehat{p} + \frac{h}{2}\widehat{q} + \overline{R}$

$\overline{B_2} = \frac{h}{2}\widehat{p} - \frac{h}{2}\widehat{q} + \overline{R}$

$\overline{B_3} = -\frac{h}{2}\widehat{p} - \frac{h}{2}\widehat{q} + \overline{R}$

$\overline{B_4} = -\frac{h}{2}\widehat{p} + \frac{h}{2}\widehat{q} + \overline{R}$

Note that the ratio of the effective area of dipole B to the effective area of coil B is $\frac{B}{h^2}$.

A point on coil B is given by:

$(p, q) = p\,\widehat{p} + q\,\widehat{q} + \overline{R}$

The distance from a point on coil A to a point on coil B is:

$r_{AB} = |\ \overline{s_A} - p\,\widehat{p} - q\,\widehat{q}\ |$

where the vector from center of coil B to the point on coil A is:

$\overline{s_A} = \mu\,\widehat{u} + \nu\,\widehat{v} - \overline{R}$

Expand the integral around circuit B in equation A.2 by explicitly going around circuit B from point $\overline{B_4}$ through points $\overline{B_1}, \overline{B_2}, \overline{B_3}$, and back to point $\overline{B_4}$:

$$L_m = \frac{\mu_o}{4\pi} \lim_{h\to 0} \frac{B}{h^2} \oint_A \int_{-\frac{h}{2}}^{+\frac{h}{2}} \left(\frac{\widehat{p}\,db}{|\overline{s_A} - bh\widehat{p} - \frac{h}{2}\widehat{q}|} - \frac{\widehat{q}\,db}{|\overline{s_A} - \frac{h}{2}\widehat{p} + bh\widehat{q}|} - \frac{\widehat{p}\,db}{|\overline{s_A} + bh\widehat{p} + \frac{h}{2}\widehat{q}|} + \frac{\widehat{q}\,db}{|\overline{s_A} + \frac{h}{2}\widehat{p} - bh\widehat{q}|} \right) \cdot \overline{ds_A} \tag{A.3}$$

Define g to be the inverse of the distance from a point on coil A to a point on coil B:

$$g = \frac{1}{r_{AB}} = \frac{1}{|\ \overline{s_A} - p\,\widehat{p} - q\,\widehat{q}\ |} \tag{A.4}$$

Calculate some directional partial derivatives along the $\widehat{p}$ and $\widehat{q}$ axes, remembering that $\overline{s_A}$ is independent of p and q:

$$\left[\frac{\partial g}{\partial p}\right]_{p=0}^{q=\frac{h}{2}} = \frac{\partial}{\partial p}\left[\frac{1}{|\overline{s_A} - p\widehat{p} - q\widehat{q}|}\right]_{p=0,q=\frac{h}{2}} = \lim_{\epsilon\to 0} \frac{1}{2b\epsilon}\left(\frac{1}{|\overline{s_A} - b\epsilon\widehat{p} - \frac{h}{2}\widehat{q}|} - \frac{1}{|\overline{s_A} + b\epsilon\widehat{p} - \frac{h}{2}\widehat{q}|} \right) \tag{A.5}$$

$$\left[\frac{\partial g}{\partial q}\right]_{p=-bh}^{q=0} = \frac{\partial}{\partial q}\left[\frac{1}{|\overline{s_A} - p\widehat{p} - q\widehat{q}|}\right]_{p=-bh,q=0} = \lim_{\epsilon\to 0} \frac{1}{\epsilon}\left(\frac{1}{|\overline{s_A} + bh\widehat{p} - \frac{\epsilon}{2}\widehat{q}|} - \frac{1}{|\overline{s_A} + bh\widehat{p} + \frac{\epsilon}{2}\widehat{q}|} \right) \tag{A.6}$$

$$\left[\frac{\partial g}{\partial p}\right]_{p=0}^{q=-bh} = \frac{\partial}{\partial p}\left[\frac{1}{|\overline{s_A} - p\widehat{p} - q\widehat{q}|}\right]_{p=0,q=-bh} = \lim_{\epsilon\to 0} \frac{1}{\epsilon}\left(\frac{1}{|\overline{s_A} - \frac{\epsilon}{2}\widehat{p} + bh\widehat{q}|} - \frac{1}{|\overline{s_A} + \frac{\epsilon}{2}\widehat{p} + bh\widehat{q}|} \right) \tag{A.7}$$

$$\left[\frac{\partial g}{\partial q}\right]_{p=-\frac{h}{2}}^{q=0} = \frac{\partial}{\partial q}\left[\frac{1}{|\overline{s_A} - p\widehat{p} - q\widehat{q}|}\right]_{p=-\frac{h}{2},q=0} = \lim_{\epsilon\to 0}\frac{1}{2b\epsilon}\left(\frac{1}{|\overline{s_A} + \frac{h}{2}\widehat{p} - b\epsilon\widehat{q}|} - \frac{1}{|\overline{s_A} + \frac{h}{2}\widehat{p} + b\epsilon\widehat{q}|}\right) \quad (A.8)$$

Set $\epsilon = h$, and substitute into equation A.3:

$$L_m = \frac{\mu_o B}{4\pi}\lim_{h\to 0}\frac{1}{h}\oint_A\int_{-\frac{h}{2}}^{+\frac{h}{2}}\left(2b\widehat{p}\left[\frac{\partial g}{\partial p}\right]_{p=0}^{q=\frac{h}{2}} + \widehat{p}\left[\frac{\partial g}{\partial q}\right]_{p=-bh}^{q=0} - \widehat{q}\left[\frac{\partial g}{\partial p}\right]_{p=0}^{q=-bh} + 2b\widehat{q}\left[\frac{\partial g}{\partial q}\right]_{p=-\frac{h}{2}}^{q=0}\right) db\cdot d\overline{s_A} \quad (A.9)$$

Perform the integration over b:

$$L_m = \frac{\mu_o B}{4\pi}\lim_{h\to 0}\oint_A\left(\widehat{p}\left[\frac{\partial g}{\partial q}\right]_{p=-bh}^{q=0} - \widehat{q}\left[\frac{\partial g}{\partial p}\right]_{p=0}^{q=-bh}\right)\cdot d\overline{s_A} \quad (A.10)$$

Expand the integral around circuit A in equation A.10 by explicitly going around circuit A from point $\overline{A_4}$ through points $\overline{A_1}$, $\overline{A_2}$, $\overline{A_3}$, and back to point $\overline{A_4}$:

$$L_m = \frac{\mu_o B}{4\pi}\lim_{h\to 0}(I_1 + I_2 + I_3 + I_4) \quad (A.11)$$

where:

$$I_1 = \frac{A}{h^2}\int_{-\frac{h}{2}}^{\frac{h}{2}}\left(\widehat{p}\left[\frac{\partial g}{\partial q}\right]_{p=q=0} - \widehat{q}\left[\frac{\partial g}{\partial p}\right]_{p=q=0}\right)_{\nu=\frac{h}{2}}\cdot\widehat{u}\,d\mu \quad (A.12)$$

$$I_2 = -\frac{A}{h^2}\int_{-\frac{h}{2}}^{\frac{h}{2}}\left(\widehat{p}\left[\frac{\partial g}{\partial q}\right]_{p=q=0} - \widehat{q}\left[\frac{\partial g}{\partial p}\right]_{p=q=0}\right)_{\mu=\frac{h}{2}}\cdot\widehat{v}\,d\nu \quad (A.13)$$

$$I_3 = -\frac{A}{h^2}\int_{-\frac{h}{2}}^{\frac{h}{2}}\left(\widehat{p}\left[\frac{\partial g}{\partial q}\right]_{p=q=0} - \widehat{q}\left[\frac{\partial g}{\partial p}\right]_{p=q=0}\right)_{\nu=-\frac{h}{2}}\cdot\widehat{u}\,d\mu \quad (A.14)$$

$$I_4 = \frac{A}{h^2}\int_{-\frac{h}{2}}^{\frac{h}{2}}\left(\widehat{p}\left[\frac{\partial g}{\partial q}\right]_{p=q=0} - \widehat{q}\left[\frac{\partial g}{\partial p}\right]_{p=q=0}\right)_{\mu=-\frac{h}{2}}\cdot\widehat{v}\,d\nu \quad (A.15)$$

Rearrange to separate the four dotproducts:

$$L_m = \frac{\mu_o AB}{4\pi}\lim_{h\to 0}(I_{pu}\,\widehat{p}\cdot\widehat{u} - I_{qu}\,\widehat{q}\cdot\widehat{u} - I_{pv}\,\widehat{p}\cdot\widehat{v} + I_{qv}\,\widehat{q}\cdot\widehat{v}) \quad (A.16)$$

where:

$$I_{pu} = \frac{1}{h^2}\int_{-\frac{h}{2}}^{\frac{h}{2}}\left(\left[\frac{\partial g}{\partial q}\right]_{p=q=0}^{\nu=\frac{h}{2}} - \left[\frac{\partial g}{\partial q}\right]_{p=q=0}^{\nu=-\frac{h}{2}}\right)d\mu \quad (A.17)$$

$$I_{qu} = \frac{1}{h^2} \int_{-\frac{h}{2}}^{\frac{h}{2}} \left(\left[\frac{\partial g}{\partial p} \right]_{p=q=0}^{\nu=\frac{h}{2}} - \left[\frac{\partial g}{\partial p} \right]_{p=q=0}^{\nu=-\frac{h}{2}} \right) d\mu \tag{A.18}$$

$$I_{pv} = \frac{1}{h^2} \int_{-\frac{h}{2}}^{\frac{h}{2}} \left(\left[\frac{\partial g}{\partial q} \right]_{p=q=0}^{\mu=\frac{h}{2}} - \left[\frac{\partial g}{\partial q} \right]_{p=q=0}^{\mu=-\frac{h}{2}} \right) d\nu \tag{A.19}$$

$$I_{qv} = \frac{1}{h^2} \int_{-\frac{h}{2}}^{\frac{h}{2}} \left(\left[\frac{\partial g}{\partial p} \right]_{p=q=0}^{\mu=\frac{h}{2}} - \left[\frac{\partial g}{\partial p} \right]_{p=q=0}^{\mu=-\frac{h}{2}} \right) d\nu \tag{A.20}$$

Apply the definition of a partial derivative and the fact that h is small to equations A.17, A.18, A.19, and A.20:

$$I_{pu} = \frac{1}{h} \int_{-\frac{h}{2}}^{\frac{h}{2}} \left[\frac{\partial^2 g}{\partial \nu \partial q} \right]_{p=q=0}^{\mu=\nu=0} d\mu = \left[\frac{\partial^2 g}{\partial \nu \partial q} \right]_{p=q=0}^{\mu=\nu=0} \tag{A.21}$$

$$I_{qu} = \frac{1}{h} \int_{-\frac{h}{2}}^{\frac{h}{2}} \left[\frac{\partial^2 g}{\partial \nu \partial p} \right]_{p=q=0}^{\mu=\nu=0} d\mu = \left[\frac{\partial^2 g}{\partial \nu \partial p} \right]_{p=q=0}^{\mu=\nu=0} \tag{A.22}$$

$$I_{pv} = \frac{1}{h} \int_{-\frac{h}{2}}^{\frac{h}{2}} \left[\frac{\partial^2 g}{\partial \mu \partial q} \right]_{p=q=0}^{\mu=\nu=0} d\nu = \left[\frac{\partial^2 g}{\partial \mu \partial q} \right]_{p=q=0}^{\mu=\nu=0} \tag{A.23}$$

$$I_{qv} = \frac{1}{h} \int_{-\frac{h}{2}}^{\frac{h}{2}} \left[\frac{\partial^2 g}{\partial \mu \partial p} \right]_{p=q=0}^{\mu=\nu=0} d\nu = \left[\frac{\partial^2 g}{\partial \mu \partial p} \right]_{p=q=0}^{\mu=\nu=0} \tag{A.24}$$

Recall that g is:

$$g = \frac{1}{r_{AB}} = \frac{1}{\mid \mu\,\widehat{u} + \nu\,\widehat{v} - \overline{R} - p\,\widehat{p} - q\,\widehat{q} \mid} = \frac{1}{\mid (\mu - u)\,\widehat{u} + (\nu - v)\,\widehat{v} - a\,\widehat{a} - p\,\widehat{p} - q\,\widehat{q} \mid} \tag{A.25}$$

Then:

$$I_{pu} = \left[\frac{\partial^2 g}{\partial \nu \partial q} \right]_{p=q=0}^{\mu=\nu=0} = - \left[\frac{\partial^2 g}{\partial v \partial q} \right]_{p=q=0}^{\mu=\nu=0} \tag{A.26}$$

$$I_{qu} = \left[\frac{\partial^2 g}{\partial \nu \partial p} \right]_{p=q=0}^{\mu=\nu=0} = - \left[\frac{\partial^2 g}{\partial v \partial p} \right]_{p=q=0}^{\mu=\nu=0} \tag{A.27}$$

$$I_{pv} = \left[\frac{\partial^2 g}{\partial \mu \partial q} \right]_{p=q=0}^{\mu=\nu=0} = - \left[\frac{\partial^2 g}{\partial u \partial q} \right]_{p=q=0}^{\mu=\nu=0} \tag{A.28}$$

$$I_{qv} = \left[\frac{\partial^2 g}{\partial \mu \partial p} \right]_{p=q=0}^{\mu=\nu=0} = - \left[\frac{\partial^2 g}{\partial u \partial p} \right]_{p=q=0}^{\mu=\nu=0} \tag{A.29}$$

Substitute into equation A.16:

$$L_m = \frac{\mu_o AB}{4\pi}\left(-\frac{\partial^2 g}{\partial v \partial q}\hat{p}\cdot\hat{u} + \frac{\partial^2 g}{\partial v \partial p}\hat{q}\cdot\hat{u} + \frac{\partial^2 g}{\partial u \partial q}\hat{p}\cdot\hat{v} - \frac{\partial^2 g}{\partial u \partial p}\hat{q}\cdot\hat{v}\right) \tag{A.30}$$

Calculate g in equation A.30 and its partial derivatives in the $\hat{u}$, $\hat{v}$, $\hat{a}$ cartesian coordinate system, where dipole A is at the origin and dipole B is at $\overline{R} = R\hat{r} = u\hat{u} + v\hat{v} + a\hat{a}$:

$g = \frac{1}{R} = \frac{1}{\sqrt{u^2+v^2+a^2}}$

$\frac{\partial g}{\partial p} = \hat{p}\cdot\hat{u}\frac{\partial g}{\partial u} + \hat{p}\cdot\hat{v}\frac{\partial g}{\partial v} + \hat{p}\cdot\hat{a}\frac{\partial g}{\partial a} = -g^3\left(\hat{p}\cdot\hat{u}u + \hat{p}\cdot\hat{v}v + \hat{p}\cdot\hat{a}a\right)$

$\frac{\partial^2 g}{\partial u \partial p} = 3g^5\left(\hat{p}\cdot\hat{u}u + \hat{p}\cdot\hat{v}v + \hat{p}\cdot\hat{a}a\right)u - g^3\hat{p}\cdot\hat{u}$

$\frac{\partial^2 g}{\partial v \partial p} = 3g^5\left(\hat{p}\cdot\hat{u}u + \hat{p}\cdot\hat{v}v + \hat{p}\cdot\hat{a}a\right)v - g^3\hat{p}\cdot\hat{v}$

$\frac{\partial g}{\partial q} = \hat{q}\cdot\hat{u}\frac{\partial g}{\partial u} + \hat{q}\cdot\hat{v}\frac{\partial g}{\partial v} + \hat{q}\cdot\hat{a}\frac{\partial g}{\partial a} = -g^3\left(\hat{q}\cdot\hat{u}u + \hat{q}\cdot\hat{v}v + \hat{q}\cdot\hat{a}a\right)$

$\frac{\partial^2 g}{\partial u \partial q} = 3g^5\left(\hat{q}\cdot\hat{u}u + \hat{q}\cdot\hat{v}v + \hat{q}\cdot\hat{a}a\right)u - g^3\hat{q}\cdot\hat{u}$

$\frac{\partial^2 g}{\partial v \partial q} = 3g^5\left(\hat{q}\cdot\hat{u}u + \hat{q}\cdot\hat{v}v + \hat{q}\cdot\hat{a}a\right)v - g^3\hat{q}\cdot\hat{v}$

Substitute into equation A.30:

$$L_m = \frac{\mu_o AB}{4\pi\, R^3}\left(\frac{-3}{R^2}F - T\right) \tag{A.31}$$

where

$T = 2\left((\hat{p}\cdot\hat{v})(\hat{q}\cdot\hat{u}) - (\hat{p}\cdot\hat{u})(\hat{q}\cdot\hat{v})\right) = -2(\hat{a}\cdot\hat{b})$

and

$F = (\hat{q}\cdot\hat{u}u + \hat{q}\cdot\hat{v}v + \hat{q}\cdot\hat{a}a)\,\hat{p}\cdot\hat{u}\,v - (\hat{p}\cdot\hat{u}u + \hat{p}\cdot\hat{v}v + \hat{p}\cdot\hat{a}a)\,\hat{q}\cdot\hat{u}\,v - (\hat{q}\cdot\hat{u}u + \hat{q}\cdot\hat{v}v + \hat{q}\cdot\hat{a}a)\,\hat{p}\cdot\hat{v}\,u + (\hat{p}\cdot\hat{u}u + \hat{p}\cdot\hat{v}v + \hat{p}\cdot\hat{a}a)\,\hat{q}\cdot\hat{v}\,u$

Expand F:

$F = (\hat{q}\cdot\hat{u})(\hat{p}\cdot\hat{u})\,vu + (\hat{q}\cdot\hat{v})(\hat{p}\cdot\hat{u})\,v^2 + (\hat{q}\cdot\hat{a})(\hat{p}\cdot\hat{u})\,va - (\hat{p}\cdot\hat{u})(\hat{q}\cdot\hat{u})\,vu - (\hat{p}\cdot\hat{v})(\hat{q}\cdot\hat{u})\,v^2 - (\hat{p}\cdot\hat{a})(\hat{q}\cdot\hat{u})\,va - (\hat{q}\cdot\hat{u})(\hat{p}\cdot\hat{v})\,u^2 - (\hat{q}\cdot\hat{v})(\hat{p}\cdot\hat{v})\,uv - (\hat{q}\cdot\hat{a})(\hat{p}\cdot\hat{v})\,ua + (\hat{p}\cdot\hat{u})(\hat{q}\cdot\hat{v})\,u^2 + (\hat{p}\cdot\hat{v})(\hat{q}\cdot\hat{v})\,uv + (\hat{p}\cdot\hat{a})(\hat{q}\cdot\hat{v})\,ua$

Rearrange terms:

$F = +(\hat{q}\cdot\hat{a})(\hat{p}\cdot\hat{u})\,va - (\hat{p}\cdot\hat{a})(\hat{q}\cdot\hat{u})\,va + (\hat{p}\cdot\hat{a})(\hat{q}\cdot\hat{v})\,ua - (\hat{q}\cdot\hat{a})(\hat{p}\cdot\hat{v})\,ua - (\hat{q}\cdot\hat{u})(\hat{p}\cdot\hat{v})\,u^2 + (\hat{p}\cdot\hat{u})(\hat{q}\cdot\hat{v})\,u^2 + (\hat{q}\cdot\hat{v})(\hat{p}\cdot\hat{u})\,v^2 - (\hat{p}\cdot\hat{v})(\hat{q}\cdot\hat{u})\,v^2 + (\hat{p}\cdot\hat{u})(\hat{q}\cdot\hat{u})\,vu - (\hat{p}\cdot\hat{u})(\hat{q}\cdot\hat{u})\,vu - (\hat{p}\cdot\hat{v})(\hat{q}\cdot\hat{v})\,uv + (\hat{p}\cdot\hat{v})(\hat{q}\cdot\hat{v})\,uv$

Eliminate cancelling terms at end:

$F = +(\hat{q}\cdot\hat{a})(\hat{p}\cdot\hat{u})\,va - (\hat{p}\cdot\hat{a})(\hat{q}\cdot\hat{u})\,va + (\hat{p}\cdot\hat{a})(\hat{q}\cdot\hat{v})\,ua - (\hat{q}\cdot\hat{a})(\hat{p}\cdot\hat{v})\,ua - (\hat{q}\cdot\hat{u})(\hat{p}\cdot\hat{v})\,u^2 + (\hat{p}\cdot\hat{u})(\hat{q}\cdot\hat{v})\,u^2 + (\hat{q}\cdot\hat{v})(\hat{p}\cdot\hat{u})\,v^2 - (\hat{p}\cdot\hat{v})(\hat{q}\cdot\hat{u})\,v^2$

Insert new cancelling terms at end:

$F = +(\hat{q}\cdot\hat{a})(\hat{p}\cdot\hat{u})\,va - (\hat{p}\cdot\hat{a})(\hat{q}\cdot\hat{u})\,va + (\hat{p}\cdot\hat{a})(\hat{q}\cdot\hat{v})\,ua - (\hat{q}\cdot\hat{a})(\hat{p}\cdot\hat{v})\,ua - (\hat{q}\cdot\hat{u})(\hat{p}\cdot\hat{v})\,u^2 + (\hat{p}\cdot\hat{u})(\hat{q}\cdot\hat{v})\,u^2 + (\hat{q}\cdot\hat{v})(\hat{p}\cdot\hat{u})\,v^2 - (\hat{p}\cdot\hat{v})(\hat{q}\cdot\hat{u})\,v^2 - (\hat{p}\cdot\hat{u})(\hat{q}\cdot\hat{v})\,a^2 + (\hat{p}\cdot\hat{v})(\hat{q}\cdot\hat{u})\,a^2 + (\hat{p}\cdot\hat{u})(\hat{q}\cdot\hat{v})\,a^2 - (\hat{p}\cdot\hat{v})(\hat{q}\cdot\hat{u})\,a^2$

Since $\hat{a} = \hat{u}\times\hat{v}$ and $\hat{b} = \hat{p}\times\hat{q}$, we have:

$-\hat{a}\cdot\hat{b} = (\hat{p}\cdot\hat{v})(\hat{q}\cdot\hat{u}) - (\hat{p}\cdot\hat{u})(\hat{q}\cdot\hat{v})$

Apply this equality in the u^2 and v^2 terms in F:

$F = +(\hat{p}\cdot\hat{u})(\hat{q}\cdot\hat{a})\,va - (\hat{p}\cdot\hat{a})(\hat{q}\cdot\hat{u})\,va - (\hat{p}\cdot\hat{v})(\hat{q}\cdot\hat{a})\,ua + (\hat{p}\cdot\hat{a})(\hat{q}\cdot\hat{v})\,ua + (\hat{a}\cdot\hat{b})\,u^2 + (\hat{a}\cdot\hat{b})\,v^2 - (\hat{p}\cdot\hat{u})(\hat{q}\cdot\hat{v})\,a^2 + (\hat{p}\cdot\hat{v})(\hat{q}\cdot\hat{u})\,a^2 + (\hat{a}\cdot\hat{b})\,a^2$

Rearrange terms:

$F = -(\hat{p}\cdot\hat{a})(\hat{q}\cdot\hat{u})\,va + (\hat{p}\cdot\hat{u})(\hat{q}\cdot\hat{a})\,va - (\hat{p}\cdot\hat{u})(\hat{q}\cdot\hat{v})\,a^2 + (\hat{p}\cdot\hat{v})(\hat{q}\cdot\hat{u})\,a^2 - (\hat{p}\cdot\hat{v})(\hat{q}\cdot\hat{a})\,ua + (\hat{p}\cdot\hat{a})(\hat{q}\cdot\hat{v})\,ua + (\hat{a}\cdot\hat{b})\,u^2 + (\hat{a}\cdot\hat{b})\,v^2 + (\hat{a}\cdot\hat{b})\,a^2$

Recall that $R^2 = u^2 + v^2 + a^2$, and substitute for last three terms:

$F = -(\hat{p}\cdot\hat{a})(\hat{q}\cdot\hat{u})\, va + (\hat{p}\cdot\hat{u})(\hat{q}\cdot\hat{a})\, va - (\hat{p}\cdot\hat{u})(\hat{q}\cdot\hat{v})\, a^2 + (\hat{p}\cdot\hat{v})(\hat{q}\cdot\hat{u})\, a^2 - (\hat{p}\cdot\hat{v})(\hat{q}\cdot\hat{a})\, ua + (\hat{p}\cdot\hat{a})(\hat{q}\cdot\hat{v})\, ua + (\hat{a}\cdot\hat{b})\, R^2$

Since $\hat{a} = \hat{u}\times\hat{v}$ and $\hat{b} = \hat{p}\times\hat{q}$, we have:

$\hat{a}\cdot\hat{r} = \frac{a}{R}$

and

$(\hat{a}\cdot\hat{r})(\hat{b}\cdot\hat{r}) = \frac{(\widehat{p\cdot a})(\widehat{q\cdot u})\, va - (\widehat{p\cdot u})(\widehat{q\cdot a})\, va + (\widehat{p\cdot u})(\widehat{q\cdot v})\, a^2 - (\widehat{p\cdot v})(\widehat{q\cdot u})\, a^2 + (\widehat{p\cdot v})(\widehat{q\cdot a})\, ua - (\widehat{p\cdot a})(\widehat{q\cdot v})\, ua}{R^2}$

Then:

$F = -(\hat{a}\cdot\hat{r})(\hat{b}\cdot\hat{r})\, R^2 + (\hat{a}\cdot\hat{b})\, R^2$

Substitute F and T into equation A.31:

$$L_m = \frac{\mu_o AB}{4\pi\, R^3}\left(3(\hat{a}\cdot\hat{r})(\hat{b}\cdot\hat{r}) - 3(\hat{a}\cdot\hat{b}) + 2(\hat{a}\cdot\hat{b})\right) \tag{A.32}$$

Simplify to get result A.1:

$$L_m = \frac{\mu_o AB}{4\pi\, R^3}\left(3(\hat{a}\cdot\hat{r})(\hat{r}\cdot\hat{b}) - (\hat{a}\cdot\hat{b})\right) \tag{A.33}$$

We can write equation A.33 using a dyadic operator:

$$L_m = \frac{\mu_o AB}{4\pi\, R^3}\,\hat{a}\cdot\mathcal{D}\cdot\hat{b} \tag{A.34}$$

If the vectors $\hat{a}$, $\hat{b}$, and $\hat{r}$ are three-element row vectors, then the dyadic operator $\mathcal{D}$ can be written as a 3x3 matrix, and the dot products are matrix multiplies:

$$L_m = \frac{\mu_o AB}{4\pi\, R^3}\,\hat{a}\,\mathcal{D}\,\hat{b}^t \tag{A.35}$$

where $\mathcal{D}$ is given by ($\mathcal{I}$ is the 3x3 identity matrix):

$$\mathcal{D} = 3\hat{r}^t\hat{r} - \mathcal{I} \tag{A.36}$$

Appendix B

Derivation of orthogonal-dipole seed calculation

Given the dipole model of mutual inductances L_{tr} as functions of position $\overline{R}$ and orientation $\mathcal{O}$, we calculate $\overline{R}$ and $\mathcal{O}$ in terms of L_{tr}.

To use the result as a P&O algorithm seed, we calculate estimated values of $\overline{R}$ and $\mathcal{O}$ from measured values of L_{tr}.

We make the dipole approximation, that the dimensions of each coil are small compared to the distance between coils.

We have three equal-effective-area colocated orthogonal dipole transmitter coils and three equal-effective-area colocated orthogonal dipole receiver coils. We derived the mutual inductance of two dipoles in Appendix A, so we start with a 3x3 mutual-inductance matrix:

$$L_{tr} = \frac{\mu_o A_t A_r}{4\pi R^3}\left(3(\widehat{t}\cdot\widehat{R})(\widehat{R}\cdot\widehat{r}) - (\widehat{t}\cdot\widehat{r})\right) \tag{B.1}$$

where
$t =$ transmitter coil 0,1, or 2
$r =$ receiver coil 0,1, or 2
$\mu_o = 4\pi 10^{-7} henries/meter$
$R =$ magnitude of distance from dipole t to dipole r
$R =$ unit vector pointing from dipole t to dipole r
$A_t =$ magnitude of transmitter dipole effective area independent of t
$A_r =$ magnitude of receiver dipole effective area independent of r
$\widehat{t} =$ unit vector pointing in direction of dipole t effective area
$\widehat{r} =$ unit vector pointing in direction of dipole r effective area
The transmitter effective area vector $\overline{A_t}$ is:
$\overline{A_t} = A_t\widehat{t}$
The receiver effective area vector $\overline{A_r}$ is:
$\overline{A_r} = A_r\widehat{r}$
Define the cartesian coordinate system:
$\widehat{X} = (1,0,0)$ unit vector in $+X$ direction
$\widehat{Y} = (0,1,0)$ unit vector in $+Y$ direction
$\widehat{Z} = (0,0,1)$ unit vector in $+Z$ direction

The receiver position vector $\overline{R}$ is independent of t and r, and has cartesian coordinates R_x, R_y, and R_z:

$\overline{R} = R\widehat{R} = R_x\widehat{X} + R_y\widehat{Y} + R_z\widehat{Z}$

We have three transmitter dipoles, 0, 1 and 2, which are centered on the origin, orthogonal to each other, all the same effective area At, and pointing along the X, Y, and Z axes respectively:

When $t = 0$, $\widehat{t} = \widehat{X}$.

When $t = 1$, $\widehat{t} = \widehat{Y}$.

When $t = 2$, $\widehat{t} = \widehat{Z}$.

Define a rotated cartesian coordinate system for the receiver dipoles, to represent the receiver orientation:

$\widehat{X_r}$ = unit vector in rotated $+X$ direction

$\widehat{Y_r}$ = unit vector in rotated $+Y$ direction

$\widehat{Z_r}$ = unit vector in rotated $+Z$ direction

We represent the receiver orientation as a normalized rotation quaternion $\mathcal{O}$. Then we calculate $\widehat{X_r}$, $\widehat{Y_r}$, and $\widehat{Z_r}$ by representating each vector as a pure-imaginary quaternion ($\mathcal{O}'$ is the quaternion complement of $\mathcal{O}$):

$(0, \widehat{X_r}) = \mathcal{O}(0, 1, 0, 0)\mathcal{O}'$

$(0, \widehat{Y_r}) = \mathcal{O}(0, 0, 1, 0)\mathcal{O}'$

$(0, \widehat{Z_r}) = \mathcal{O}(0, 0, 0, 1)\mathcal{O}'$

We have three receiver dipoles, 0, 1 and 2, which are centered on the point R, orthogonal to each other, all the same effective area Ar, and pointing along the X_r, Y_r, and Z_r axes respectively:

When $r = 0$, $\widehat{r} = \widehat{X_r}$.

When $r = 1$, $\widehat{r} = \widehat{Y_r}$.

When $r = 2$, $\widehat{r} = \widehat{Z_r}$.

Define K, which is independent of $\overline{R}$, $\mathcal{O}$, and Ltr:

$K = \frac{\mu_o A_t A_r}{4\pi}$

Now B.1 becomes:

$$L_{tr} = \frac{K}{R^3}\left(3(\widehat{t}\cdot\widehat{R})(\widehat{r}\cdot\widehat{R}) - (\widehat{t}\cdot\widehat{r})\right) \tag{B.2}$$

L_t^2 is independent of $\mathcal{O}$. We use this fact to solve for $\overline{R}$.

From B.2, the square of the mutual inductance of one transmitter dipole and one receiver dipole is:

$$L_{tr}^2 = \frac{K^2}{R^6}\left(9(\widehat{t}\cdot\widehat{R})^2(\widehat{r}\cdot\widehat{R})^2 - 6((\widehat{t}\cdot\widehat{R})(\widehat{r}\cdot\widehat{R})(\widehat{t}\cdot\widehat{r})) + (\widehat{t}\cdot\widehat{r})^2\right) \tag{B.3}$$

Then, the sum of the squares of the mutual inductances of one transmitter dipole and the three receiver dipoles, is:

$$L_t^2 = \frac{K^2}{R^6}\left(3(\widehat{t}\cdot\widehat{R})^2 + 1\right) \tag{B.4}$$

For transmitter coil 0:

$$L_0^2 = \frac{K^2}{R^6}\left(3(\frac{Rx}{R})^2 + 1\right) \tag{B.5}$$

For transmitter coil 1:

$$L_1^2 = \frac{K^2}{R^6}\left(3(\frac{Ry}{R})^2 + 1\right) \tag{B.6}$$

For transmitter coil 2:

$$L_2^2 = \frac{K^2}{R^6}\left(3(\frac{Rz}{R})^2 + 1\right) \tag{B.7}$$

The sum of the squares of all nine mutual inductances is then:

$$L_{tot}^2 = 6\frac{K^2}{R^6} \tag{B.8}$$

Solve B.8 for R:

$$R = \left(\frac{6K^2}{L_{tot}^2}\right)^{\frac{1}{6}} \tag{B.9}$$

Solve B.5, B.6, and B.7 for Rx^2, Ry^2 and Rz^2 respectively, and use B.8 and B.9 to eliminate R:

$$R_x^2 = \left(2\frac{L_0^2}{L_{tot}^2} - \frac{1}{3}\right)\left(\frac{6K^2}{L_{tot}^2}\right)^{\frac{1}{3}} \tag{B.10}$$

$$R_y^2 = \left(2\frac{L_1^2}{L_{tot}^2} - \frac{1}{3}\right)\left(\frac{6K^2}{L_{tot}^2}\right)^{\frac{1}{3}} \tag{B.11}$$

$$R_z^2 = \left(2\frac{L_2^2}{L_{tot}^2} - \frac{1}{3}\right)\left(\frac{6K^2}{L_{tot}^2}\right)^{\frac{1}{3}} \tag{B.12}$$

Now we take the square roots of these three quantities:

$$R_x = (+or-)\sqrt{\left(2\frac{L_0^2}{L_{tot}^2} - \frac{1}{3}\right)\left(\frac{6K^2}{L_{tot}^2}\right)^{\frac{1}{3}}} \tag{B.13}$$

$$R_y = (+or-)\sqrt{\left(2\frac{L_0^2}{L_{tot}^2} - \frac{1}{3}\right)\left(\frac{6K^2}{L_{tot}^2}\right)^{\frac{1}{3}}} \tag{B.14}$$

$$R_z = (+or-)\sqrt{\left(2\frac{L_0^2}{L_{tot}^2} - \frac{1}{3}\right)\left(\frac{6K^2}{L_{tot}^2}\right)^{\frac{1}{3}}} \tag{B.15}$$

The signs of Rx, Ry, and Rz are determined from the signs of products of the mutual inductances.

Due to the symmetries of the fields, positions $\overline{+R}$ and $\overline{-R}$, for the same orientation $\mathcal{O}$, give the same mutual inductances, and cannot be distinguished by calculation. In the literature, this is called the hemisphere ambiguity.

To resolve this ambiguity, we choose to use the $+X$ hemisphere, where R_x is always positive.

Form sums of products of mutual inductances L_{tr}:

$$S_{01} = L_{00}L_{10} + L_{01}L_{11} + L_{02}L_{12} \tag{B.16}$$

$$S_{20} = L_{20}L_{00} + L_{21}L_{01} + L_{22}L_{02} \tag{B.17}$$

Recall that the signum or sign function is defined to be:
$sgn(x) = +1 for\{x \geq 0\}, -1 for\{x < 0\}$
Then the signs of Rx, Ry, and Rz are given by:
$sgn(R_x) = +1$
$sgn(R_y) = sgn(S_{01})$
$sgn(R_z) = sgn(S_{20})$

Then the position $\overline{R}$ cartesian components are calculated in terms of the mutual inductances L_{tr} by:

$$L_0^2 = L_{00}L_{00} + L_{01}L_{01} + L_{02}L_{02} \tag{B.18}$$

$$L_1^2 = L_{10}L_{10} + L_{11}L_{11} + L_{12}L_{12} \tag{B.19}$$

$$L_2^2 = L_{20}L_{20} + L_{21}L_{21} + L_{22}L_{22} \tag{B.20}$$

$$L_{tot}^2 = L_0^2 + L_1^2 + L_2^2 \tag{B.21}$$

$$R_x = +\sqrt{\left(2\frac{L_0^2}{L_{tot}^2} - \frac{1}{3}\right)\left(\frac{6K^2}{L_{tot}^2}\right)^{\frac{1}{3}}} \tag{B.22}$$

$$R_y = sgn(L_{00}L_{10} + L_{01}L_{11} + L_{02}L_{12})\sqrt{\left(2\frac{L_0^2}{L_{tot}^2} - \frac{1}{3}\right)\left(\frac{6K^2}{L_{tot}^2}\right)^{\frac{1}{3}}} \tag{B.23}$$

$$R_z = sgn(L_{20}L_{00} + L_{21}L_{01} + L_{22}L_{02})\sqrt{\left(2\frac{L_0^2}{L_{tot}^2} - \frac{1}{3}\right)\left(\frac{6K^2}{L_{tot}^2}\right)^{\frac{1}{3}}} \tag{B.24}$$

As R_x, R_y, and/or R_z approach zero, B.16 and/or B.17 become numerically unstable due to small mutual inductances, and B.22, B.23, and B.24 become numerically unstable due to the square roots: Errors in the mutual inductances can cause large errors, wrong signs, or even cause a close-to-zero result to become imaginary.

Since the transmitters are colocated dipoles, and colocated dipole moments combine the same way vectors do, synthetic mutual-inductance matrices can be calculated for transmitter dipole triads with other orientations than along the X, Y, and Z axes. This solves the numerical instability of the above solution.

If the calculated position is close to the X, Y, or Z axis (and hence is numerically unstable), we can: mathematically rotate the coordinate system to move the position far from the axes, calculate the position in the rotated coordinate system (this calculation is numerically stable), then derotate the result to the original coordinate system.

In detail, the rotate, calculate, derotate algorithm is as follows:

Calculate unrotated position $\overline{R^u} = \left(R_x^u, R_y^u, R_z^u\right)$ by substituting the measured mutual inductances L_{tr} in B.22, B.23, and B.24:

$$R_x^u = +\sqrt{\left(2\frac{L_0^2}{L_{tot}^2} - \frac{1}{3}\right)\left(\frac{6K^2}{L_{tot}^2}\right)^{\frac{1}{3}}} \tag{B.25}$$

$$R_y^u = sgn(L_{00}L_{10} + L_{01}L_{11} + L_{02}L_{12})\sqrt{\left(2\frac{L_0^2}{L_{tot}^2} - \frac{1}{3}\right)\left(\frac{6K^2}{L_{tot}^2}\right)^{\frac{1}{3}}} \tag{B.26}$$

$$R_z^u = sgn(L_{20}L_{00} + L_{21}L_{01} + L_{22}L_{02})\sqrt{\left(2\frac{L_0^2}{L_{tot}^2} - \frac{1}{3}\right)\left(\frac{6K^2}{L_{tot}^2}\right)^{\frac{1}{3}}} \tag{B.27}$$

If any of R_x^u, R_y^u, or R_z^u come out imaginary, make that value or those values zero.

Note that the sign for a small-value R_x^u, R_y^u, or R_z^u may be wrong, but this causes only a small position error, which error will be removed in following steps.

Calculate a unit vector $\widehat{R^u}$in the direction of $\overline{R^u}$:

$$\widehat{R^u} = \frac{\overline{R^u}}{\sqrt{((R^u_x)^2 + (R^u_y)^2 + (R^u_z)^2)}} \tag{B.28}$$

We next calculate a unit vector $\widehat{S^u}$ in the $\left(sgn(R^u_x), sgn(R^u_y), sgn(R^u_z)\right)$ direction:

$$\widehat{S^u} = \left(\frac{sgn(R^u_x)}{\sqrt{3}}, \frac{sgn(R^u_y)}{\sqrt{3}}, \frac{sgn(R^u_z)}{\sqrt{3}}\right) \tag{B.29}$$

Note that S^u is far from the X, Y, and Z axes, and is always in the $+X$ hemisphere.

Note that our choice of $\widehat{S^u}$ means that the dot product $\widehat{R^u} \cdot \widehat{S^u}$ can never be negative.

Note that $\widehat{R^u} \cdot \widehat{S^u}$ analytically can never be greater than unity. Numerical errors can result in a value slightly greater than unity, but such errors do no harm here.

If $\widehat{R^u} \cdot \widehat{S^u}$ is greater than or equal to some threshold (we use 0.9 to 0.95), then the position estimate is numerically stable enough to be accurate, so the position is given by:

$$\overline{R} = \overline{R^u} \tag{B.30}$$

and we skip equations B.31 through B.45 below.

If the dot product is less than the threshold, then the position estimate is numerically unreliable, and we must improve the numerical stability. We do this by recalculating the position in a rotated coordinate system, and then derotating the position back to the original coordinate system.

We choose to rotate $\widehat{R^u}$ into $\widehat{S^u}$. Note that the rotated position vector will be far from the X, Y, and Z axes, even if the recalculation changes the position a little.

We could use any of various vector-rotation formalisms. We choose to use normalized rotation quaternions for their numerical stability and lack of singularities.

The angle between $\widehat{R^u}$ and $\widehat{S^u}$ is:

$\theta = \arccos(\widehat{R^u} \cdot \widehat{S^u})$

The normalized rotation quaternion to rotate $\widehat{R^u}$ into $\widehat{S^u}$ is:

$$\mathcal{U} = \left(\cos\left(\frac{\theta}{2}\right), \frac{\widehat{R^u} \times \widehat{S^u}}{\cos\left(\frac{\theta}{2}\right)}\right) \tag{B.31}$$

and

$$\left(0, \widehat{S^u}\right) = \mathcal{U}\left(0, \widehat{R^u}\right)\mathcal{U}' \tag{B.32}$$

Equation B.31 is numerically unstable when $\widehat{R^u}$ and $\widehat{S^u}$ point in close to opposite directions, i.e. when $\frac{\theta}{2}$ approaches ninety degrees. This will never happen here, because $\widehat{R^u} \cdot \widehat{S^u}$ is never negative.

We treat the 3x3 matrix of mutual inductances, L_{tr}, in B.2 as a trio of column (transposed row) vectors, one for each receiver coil, where the superscript t indicates the transpose operation:

$$L_{tr} = \left(L^t_0, L^t_1, L^t_2\right) \tag{B.33}$$

To rotate the transmitter coordinates, we apply the rotation $\mathcal{U}$ to each of the three row vectors in B.33:

$$(0, L^r_0) = \mathcal{U}\,(0, L_0)\,\mathcal{U}' \tag{B.34}$$

$$(0, L_1^r) = \mathcal{U}\,(0, L_1)\,\mathcal{U}' \tag{B.35}$$

$$(0, L_2^r) = \mathcal{U}\,(0, L_2)\,\mathcal{U}' \tag{B.36}$$

The rotated mutual-inductance matrix is:

$$L_{tr}^r = \left(L_0^{rt}, L_1^{rt}, L_2^{rt}\right) \tag{B.37}$$

Calculate rotated position $\overline{R^r} = \left(R_x^r, R_y^r, R_z^r\right)$ by substituting the rotated mutual inductances L_{tr}^r in B.18 through B.24:

$$L_0^{r2} = L_{00}^r L_{00}^r + L_{01}^r L_{01}^r + L_{02}^r L_{02}^r \tag{B.38}$$

$$L_1^{r2} = L_{10}^r L_{10}^r + L_{11}^r L_{11}^r + L_{12}^r L_{12}^r \tag{B.39}$$

$$L_2^{r2} = L_{20}^r L_{20}^r + L_{21}^r L_{21}^r + L_{22}^r L_{22}^r \tag{B.40}$$

$$L_{tot}^2 = L_0^2 + L_1^2 + L_2^2 \tag{B.41}$$

$$R_x^r = +\sqrt{\left(2\frac{L_0^{r2}}{L_{tot}^{r2}} - \frac{1}{3}\right)\left(\frac{6K^2}{L_{tot}^{r2}}\right)^{\frac{1}{3}}} \tag{B.42}$$

$$R_y^r = sgn(L_{00}L_{10} + L_{01}L_{11} + L_{02}L_{12})\sqrt{\left(2\frac{L_0^2}{L_{tot}^2} - \frac{1}{3}\right)\left(\frac{6K^2}{L_{tot}^2}\right)^{\frac{1}{3}}} \tag{B.43}$$

$$R_z^r = sgn(L_{20}L_{00} + L_{21}L_{01} + L_{22}L_{02})\sqrt{\left(2\frac{L_0^2}{L_{tot}^2} - \frac{1}{3}\right)\left(\frac{6K^2}{L_{tot}^2}\right)^{\frac{1}{3}}} \tag{B.44}$$

None of R_x^r, R_y^r, or R_z^r will be small, imaginary, or otherwise numerically unstable, as the rotated position is far from the axes.

Derotate the position:

$$\left(0, \overline{R}\right) = \mathcal{U}'\left(0, \overline{R^r}\right)\mathcal{U} \tag{B.45}$$

We now have an accurate position estimate $\overline{R}$ from B.30 or B.45 .

Note that we imposed the $+X$ hemisphere restriction to force R_x to be positive, but numerical effects may generate R_x which is slightly negative. If R_x is negative, it will be only by a value within the errors due to errors in L_{tr}. Physically, this happens only when the physical X position is zero, putting the position on the hemisphere boundary.

We now determine an estimate for orientation. Recall that the receiver coil axes are defined by:

When $r = 0$, $\hat{r} = \widehat{X_r}$.

When $r = 1$, $\hat{r} = \widehat{Y_r}$.

When $r = 2$, $\hat{r} = \widehat{Z_r}$.

At zero orientation, the three receiver coil axes are aligned with the coordinate system axes:

When $r = 0$, $\widehat{r^0} = \widehat{X}$.

When $r = 1$, $\widehat{r^0} = \widehat{Y}$.

When $r = 2$, $\widehat{r^0} = \widehat{Z}$.

We represented the orientation as a normalized rotation quaternion $\mathcal{O}$:

$$\left(0, \widehat{X_r}\right) = \mathcal{O}\left(0, \widehat{X}\right)\mathcal{O}' \tag{B.46}$$

$$\left(0, \widehat{Y_r}\right) = \mathcal{O}\left(0, \hat{Y}\right)\mathcal{O}' \tag{B.47}$$

$$\left(0, \widehat{Z_r}\right) = \mathcal{O}\left(0, \hat{Z}\right)\mathcal{O}' \tag{B.48}$$

We can also represent the orientation as an orthonormal rotation matrix O, remembering that the vectors are row vectors:

$$\widehat{X_r} = \hat{X}O \tag{B.49}$$

$$\widehat{Y_r} = \hat{Y}O \tag{B.50}$$

$$\widehat{Z_r} = \hat{Z}O \tag{B.51}$$

Equations B.49, B.50, and B.51 can be combined into one matrix equation, where the superscript t indicates the transpose of a vector or matrix:

$$\left(\widehat{X_r}^t, \widehat{Y_r}^t, \widehat{Z_r}^t\right) = O^t\left(\hat{X}^t, \hat{Y}^t, \hat{Z}^t\right) \tag{B.52}$$

For most orientation calculations, the rotation-matrix representation is undesireable: Rotation matrices can easily become non-orthonormal, as nine numbers in the matrix encode three degrees of freedom.

However, rotation matrices have one advantage over rotation quaternions: Equation B.52 is easily solved for the rotation matrix, where the superscript -1 indicates the inverse of a matrix:

$$O^t = \left(\widehat{X_r}^t, \widehat{Y_r}^t, \widehat{Z_r}^t\right)\left(\hat{X}^t, \hat{Y}^t, \hat{Z}^t\right)^{-1} \tag{B.53}$$

Substituting the zero-orientation receiver unit vectors in B.2 gives L^0_{tr}, the mutual-inductance matrix at position $\overline{R}$ for zero orientation:

$$L^0_{tr} = \frac{K}{R^3}\left(3(\hat{t}\cdot\hat{R})(\widehat{r^0}\cdot\hat{R}) - (\hat{t}\cdot\widehat{r^0})\right) \tag{B.54}$$

In B.2 and B.54, the three mutual inductances for each receiver coil form a column vector, analogous to the column vectors in B.52. The measured mutual inductances, L_{tr}, thus differ from L^0_{tr} by the the receiver rotation:

$$L_{tr} = O^t L^0_{tr} \tag{B.55}$$

Solve B.55 for O:

$$O^t = L_{tr} L^{0-1}_{tr} \tag{B.56}$$

Ideally, O is an orthonormal rotation matrix describing the receiver rotation. Measurement errors and field nonidealities can easily make O deviate from orthonormality.

If matrix O is not orthonormal, we cannot convert O exactly to a rotation quaternion $\mathcal{O}$. The best we can do, is to convert to a best-fit quaternion in the least-squares sense, as described by an article entitled "Closed-form solution of absolute orientation using unit quaternions", by Berthold K.P. Horn, published in "Journal of the Optical Society of America", volume 4, April, 1987, pages 629ff, herein incorporated by reference.

Appendix C

Mutual inductance of circuits of straight-line segments

The mutual inductance, L_m, between two closed circuits is given by Neuman's formula:

$$L_m = \frac{\mu_o}{4\pi} \oint_{circuitA} \oint_{circuitB} \frac{d\overline{s_A} \cdot d\overline{s_B}}{r_{AB}} \tag{C.1}$$

where
$r_{AB} = \mid \overline{s_B} - \overline{s_A} \mid$
and
$\mu_o = 4\pi 10^{-7} henries/meter$
Here, we consider the case where both circuits are composed of straight-line segments.
Circuit A is composed of segments $A_1 \cdots A_n$.
Circuit B is composed of segments $B_1 \cdots B_m$.
Then C.1 becomes:

$$L_m = \frac{\mu_o}{4\pi} \sum_{k=1}^{n} \sum_{l=1}^{m} \int_{B_l^{start}}^{B_l^{finish}} \int_{A_k^{start}}^{A_k^{finish}} \frac{d\overline{s_A} \cdot d\overline{s_B}}{r_{AB}} \tag{C.2}$$

The integral over A_k can be evaluated analytically:

$$L_m = \frac{\mu_o}{4\pi} \sum_{k=1}^{n} \sum_{l=1}^{m} \frac{\overline{s_l} \cdot \overline{s_k}}{\sqrt{s_k^2}} \int_{b=0}^{b=1} \log\left(\frac{\sqrt{\frac{(\overline{s_k}+s_g\overline{s_b})\cdot(\overline{s_k}+s_g\overline{s_b})}{s_b^2}} + \sqrt{\frac{s_k^2}{s_b^2}} + s_g c_l}{1 + s_g c_l}\right) db \tag{C.3}$$

where
$\overline{s_k^{start}}$ = vector from origin to start of segment A_k
$\overline{s_k^{finish}}$ = vector from origin to finish of segment A_k
$\overline{s_l^{start}}$ = vector from origin to start of segment B_l
$\overline{s_l^{finish}}$ = vector from origin to finish of segment B_l
$\overline{s_k} = \overline{s_k^{finish}} - \overline{s_k^{start}}$
$\overline{s_l} = \overline{s_l^{finish}} - \overline{s_l^{start}}$
$s_k^2 = \overline{s_k} \cdot \overline{s_k}$

$s_l^2 = \overline{s_l} \cdot \overline{s_l}$
$\overline{s_s} = (\overline{s_l^{start}} - \overline{s_k^{start}})$
$s_g = +1$ if $(\overline{s_s} \cdot \overline{s_k} \geq 0)$, -1 if $(\overline{s_s} \cdot \overline{s_k} < 0)$
$\overline{s_e} = \overline{s_k^{finish}}$ if $(s_g = +1)$, $\overline{s_k^{start}}$ if $(s_g = -1)$
$\overline{s_b} = \overline{s_l^{start}} - \overline{s_e} + \overline{s_l} b$
$s_b^2 = \overline{s_b} \cdot \overline{s_b}$
$c_t = \frac{\overline{s_b} \cdot \overline{s_k}}{\sqrt{s_b^2 s_k^2}}$ = cosine of angle between $\overline{s_b}$ and $\overline{s_k}$

Equation C.3 is easily evaluated numerically by well-known numerical integration methods.

Equation C.3 combines two forms, which integrate in opposite directions along segment A_k. The two forms are combined because each form has a region of potential numerical instability due to a zero denominator.

The first region of numerical instability occurs when $\overline{s_s}$ and $\overline{s_b}$ approach being parallel. In this region, $\overline{s_s} \cdot \overline{s_b} > 0$, and $1 - c_t$ approaches zero. When $\overline{s_s}$ and $\overline{s_b}$ become exactly parallel, $1 - c_t = 0$ exactly. In this region, we choose $s_g = +1$ to avoid a zero denominator in the integral in equation C.3.

The second region of numerical instability occurs when $\overline{s_s}$ and $\overline{s_b}$ approach being antiparallel. In this region, $\overline{s_s} \cdot \overline{s_b} < 0$, and $1 + c_t$ approaches zero. When $\overline{s_s}$ and $\overline{s_b}$ become exactly parallel, $1 + c_t = 0$ exactly. In this region, we choose $s_g = -1$ to avoid a zero denominator in the integral in equation C.3.

To derive equation C.3, start with the term in equation C.1 for one pair of current segments:

$$I_{kl} = \int_{B_l^{start}}^{B_l^{finish}} \int_{A_k^{start}}^{A_k^{finish}} \frac{d\overline{s_A} \cdot d\overline{s_B}}{r_{AB}} \tag{C.4}$$

Current segments A_k and B_l are both straight lines, so it is natural to describe them in linear terms of dimensionless variables a and b respectively, where

$0 \leq a \leq 1$
$0 \leq b \leq 1$

We integrate over a symbolically first, so we will keep dependendencies on a explicit. Dependencies on b are absorbed into quantities independent of a while we integrate over a.

A point on segment A_k is one of two linear functions of a, depending on s_g:
When $s_g = +1$, then $\overline{s_A(a)} = \overline{s_k^{finish}} - \overline{s_k} a$
When $s_g = -1$, then $\overline{s_A(a)} = \overline{s_k^{start}} + \overline{s_k} a$
A point on segment B_l is a linear function of b:
$\overline{s_B(b)} = \overline{s_l^{start}} + \overline{s_l} b$
Then:
$d\overline{s_A} = -s_g \overline{s_k}\, da$
$d\overline{s_B} = \overline{s_l}\, db$
$d\overline{s_A} \cdot d\overline{s_B} = -s_g \overline{s_k} \cdot \overline{s_l}\, da\, db$
$r_{AB} = \sqrt{(\overline{s_b} + s_g \overline{s_k} a) \cdot (\overline{s_b} + s_g \overline{s_k} a)}$
Now equation C.4 becomes:

$$I_{kl} = -s_g \overline{s_k} \cdot \overline{s_l} \int_0^1 \int_{(1+s_g)/2}^{(1-s_g)/2} \frac{da}{\sqrt{(\overline{s_b} + s_g \overline{s_k} a) \cdot (\overline{s_b} + s_g \overline{s_k} a)}}\, db \tag{C.5}$$

Expand the denominator in equation C.5, separate out the coefficient of a^2, and remove the s_g dependence from the limits of integration:

$$I_{kl} = \frac{\overline{s_k} \cdot \overline{s_l}}{\sqrt{s_k^2}} \int_0^1 \int_0^1 \frac{da}{\sqrt{\frac{s_b^2}{s_k^2} + \frac{2s_g \overline{s_b} \cdot \overline{s_k}}{s_k^2} a + a^2}} \, db \tag{C.6}$$

Define two quantities which do not depend on a:

$\alpha = \frac{s_g \overline{s_b} \cdot \overline{s_k}}{s_k^2} = \sqrt{\frac{s_b^2}{s_k^2}} s_g c_t$

$\beta = \frac{s_b^2}{s_k^2} - \alpha^2 = \frac{s_b^2}{s_k^2}(1 - c_t^2)$

Note that $0 \leq \beta \leq 2$.

Define a new variable x:

$x = a + \alpha$

Now equation C.6 becomes:

$$I_{kl} = \frac{\overline{s_k} \cdot \overline{s_l}}{\sqrt{s_k^2}} \int_0^1 \int_\alpha^{\alpha+1} \frac{dx}{\sqrt{x^2 + \beta}} \, db \tag{C.7}$$

We have for $\beta \geq 0$:

$$\int \frac{dx}{\sqrt{x^2 + \beta}} = \log(x + \sqrt{x^2 + \beta}) \tag{C.8}$$

Substitute equation C.8 into equation C.7:

$$I_{kl} = \frac{\overline{s_k} \cdot \overline{s_l}}{\sqrt{s_k^2}} \int_0^1 \log\left(\frac{\alpha + 1 + \sqrt{(\alpha+1)^2 + \beta}}{\alpha + \sqrt{\alpha^2 + \beta}}\right) db \tag{C.9}$$

Substitute the definitions of α and β into equation C.9:

$$I_{kl} = \frac{\overline{s_k} \cdot \overline{s_l}}{\sqrt{s_k^2}} \int_0^1 \log\left(\frac{\sqrt{\frac{s_b^2}{s_k^2}} s_g c_t + 1 + \sqrt{\left(\sqrt{\frac{s_b^2}{s_k^2}} s_g c_t + 1\right)^2 + \frac{s_b^2}{s_k^2}(1 - c_t^2)}}{\sqrt{\frac{s_b^2}{s_k^2}} s_g c_t + \sqrt{\left(\sqrt{\frac{s_b^2}{s_k^2}} s_g c_t\right)^2 + \frac{s_b^2}{s_k^2}(1 - c_t^2)}}\right) db \tag{C.10}$$

Remove a common factor of $\sqrt{\frac{s_b^2}{s_k^2}}$ from numerator and denominator:

$$I_{kl} = \frac{\overline{s_k} \cdot \overline{s_l}}{\sqrt{s_k^2}} \int_0^1 \log\left(\frac{s_g c_t + \sqrt{\frac{s_k^2}{s_b^2}} + \sqrt{\left(s_g c_t + \sqrt{\frac{s_k^2}{s_b^2}}\right)^2 + (1 - c_t^2)}}{s_g c_t + \sqrt{(s_g c_t)^2 + (1 - c_t^2)}}\right) db \tag{C.11}$$

Simplify:

$$I_{kl} = \frac{\overline{s_k} \cdot \overline{s_l}}{\sqrt{s_k^2}} \int_0^1 \log\left(\frac{s_g c_t + \sqrt{\frac{s_k^2}{s_b^2}} + \sqrt{\frac{2 s_g c_t \sqrt{s_k^2 s_b^2} + s_k^2 + s_b^2}{s_b^2}}}{s_g c_t + 1}\right) db \tag{C.12}$$

Substitute for c_t in the last radical in the numerator:

$$I_{kl} = \frac{\overline{s_k} \cdot \overline{s_l}}{\sqrt{s_k^2}} \int_0^1 \log\left(\frac{s_g c_t + \sqrt{\frac{s_k^2}{s_b^2}} + \sqrt{\frac{2 s_g \overline{s_b} \cdot \overline{s_k} + s_k^2 + s_b^2}{s_b^2}}}{s_g c_t + 1}\right) db \tag{C.13}$$

Factor the numerator inside the last radical in the numerator:

$$I_{kl} = \frac{\overline{s_k} \cdot \overline{s_l}}{\sqrt{s_k^2}} \int_0^1 \log\left(\frac{s_g c_t + \sqrt{\frac{s_k^2}{s_b^2}} + \sqrt{\frac{(\overline{s_k} + s_g \overline{s_b}) \cdot (\overline{s_k} + s_g \overline{s_b})}{s_b^2}}}{s_g c_t + 1}\right) db \tag{C.14}$$

Substitute equation C.14 into equation C.4, and that into equation C.2:

$$L_{mn} = \frac{\mu_o}{4\pi} \sum_{k=1}^{n} \sum_{l=1}^{m} \frac{\overline{s_k} \cdot \overline{s_l}}{\sqrt{s_k^2}} \int_0^1 \log\left(\frac{s_g c_t + \sqrt{\frac{s_k^2}{s_b^2}} + \sqrt{\frac{(\overline{s_k} + s_g \overline{s_b}) \cdot (\overline{s_k} + s_g \overline{s_b})}{s_b^2}}}{s_g c_t + 1}\right) db \tag{C.15}$$

This is our result, equation C.3.